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Axisymmetric form of Kármán-Howarth equation and its limiting forms

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Abstract. Kinematics and dynamics of homogeneous axisymmetric turbulence have been derived with the assumption that the properties of the turbulence are invariant with respect to rotation about a preferred direction λ. In particular, the "axisymmetric" equivalent of Karman-Howarth "isotropic" equation is derived using Lindborg's representation of the two-point correlation tensors of homogeneous axisymmetric turbulence. When the more constraining assumption of isotropy is made, this equation reduces to the well-known Karman-Howarth equation. There are two interesting limiting forms of the axisymmetric Karman-Howarth equation: the axisymmetric form of the energy balance equation and the axisymmetric form of the vorticity balance equation.

PACS. 47.27.Ak Fundamentals – 47.10.+g General theory

1 Introduction

In the case of homogeneous turbulence when no symmetry conditions at all are imposed, the statistics of a turbulent field are somewhat difficult to study. Indeed, the most general expression of the second-order tensor for example is given by (Batchelor [1])

$$\begin{aligned} B_{ij}(\mathbf{r}) &= \langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r})\rangle \\ &= A\,\delta_{ij} \;+\; 93 \text{ terms like } B\,r_i r_j) \\ &\quad + \;(6 \text{ terms like } C\,r_i r_j) \\ &\quad + \;(3 \text{ terms like } D\,\epsilon_{ijk} r_k) \\ &\quad + \;(12 \text{ terms like } E\,\epsilon_{ikl} r_k \lambda_l r_j) \\ &\quad + \;(6 \text{ terms like } F\,\epsilon_{ikl} r_k \lambda_l \mu_j) \end{aligned}$$

where λ and μ are two extra vector arguments, $r = |\mathbf{r}|$ and the scalar coefficients A, B, ... are functions of r^2, $r_i\lambda_i$ and $r_i\mu_i$. Angular brackets denotes averaging.

Tensors of higher order are more complicated. Because of this, it is usual and practical as well as gain in simplicity, to consider fields of turbulence which satisfy certain symmetry conditions (in statistical sense). For example, the case of turbulence which has rotational symmetry about a given line or which have axial symmetry. These cases occurs frequently in practice. Besides, it frequently happens that the above general expressions are simplified by the existence of symmetry in the suffixes (for instance $\langle u_i(\mathbf{x})u_j(\mathbf{x})u_k(\mathbf{x}+\mathbf{r})\rangle$ is unchanged by interchange of the suffixes i and j). It is also possible to simplify these expressions if the field is solenoidal: the vanishing divergence gives some of the scalar functions occurring in the expression of B_{ij} in term of the others.

The simplest example of turbulence which has been widely considered is the important special case of isotropic turbulence. This case of homogeneous and isotropic turbulence is very attractive, especially for a theoretician. It was, therefore, natural to begin with this case and try to use it to exhibit some of the characteristic properties of turbulent fields. However, one must remember that the concept of isotropic turbulence is a mathematical idealization which, at best, is convenient only for approximate description of certain special types of turbulent flow. Isotropic conditions are satisfactorily fulfilled for a certain class of turbulent flows produced in laboratory wind tunnels (Monin and Yaglom (M-Y)[2]). A fundamental result for the basic dynamic equations for the correlation functions of isotropic turbulence is Karman and Howarth equation (K-H, 1938) which connect the longitudinal scalar function of second-order, $B_{\mathrm{LL}}(\mathbf{r},t)$=$\langle u_{\mathrm{L}}(\mathbf{x},t)u_{\mathrm{L}}(\mathbf{x}+\mathbf{r},t)\rangle$, to the third-order one, $B_{\mathrm{LL,L}}(\mathbf{r},t)$= $\langle u_{\mathrm{L}}^2(\mathbf{x},t)u_{\mathrm{L}}(\mathbf{x}+\mathbf{r},t)\rangle$. Many experimental work in literature ([2], p. 123) gave support to this theoretical result.

Theoretical and numerical investigations have also been devoted to K-H equation for various applications. For example, Henriksen and Lachieze [3] introduced a new approach to the calculation of the galactic mass multiplicity function based on the probability of the correlated velocity structures existing on a given spatial scale. The predictions are deduced directly from the cosmic von Karman-Howarth equation and are in good agreement with existing data.

Departing from the equations of motion of a dusty incompressible gas, Saha [4] obtained an equation which

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resembles to the K-H equation and showed that the criteria for K-H equation are also the criteria for this equation.

Recently, Chkhetiani [5] derived K-H type equations describing the evolution of the mixed correlation tensor of the velocity and the vorticity in homogeneous helical turbulence.

More recently, Gotoh [6] examined theoretically and by high resolution direct numerical simulations (DNS) the energy spectrum in the inertial and dissipation ranges in 2-D steady turbulence.

Politano and Pouquet [7] derived a von Karman-Howarth equation for magnetohydrodynamics and its consequences on the third-order longitudinal structure and correlation functions.

A type of turbulence which is next to local isotropy in order of simplicity, but which corresponds more closely to turbulent flows encountered in practice, is axisymmetric turbulence. The study of axisymmetric turbulence is particularly interesting since it is the simplest form of turbulence in which effects of anisotropy distribution among different scales and return to isotropy can be studied. It would seem profitable to examine the form that this equation (K-H) would take when only local axisymmetry, an assumption which is intermediate in severity between local homogeneity and local isotropy is adopted.

It seems appropriate to briefly review the progress that has been made on the theory of axisymmetric turbulence since Batchelor [8] first considered the second-order two-point correlation tensor using the invariant theory of Robertson [9]. This tensor can be expressed in terms of a unit vector λ in the direction of symmetry

$$\begin{aligned} B_{ij}(\mathbf{r}) &= \langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r})\rangle \\ &= r_i r_j A + \delta_{ij} B + \lambda_i \lambda_j C + (\lambda_i r_j + \lambda_j r_i) D \end{aligned} \tag{1}$$

and in terms of four scalar functions (A, B, C, D) of $r = |\mathbf{r}|$ and $\mu = \mathbf{r}\cdot\lambda/r$. Continuity gives two equations relating these four scalar functions. Since the tensor B_{ij} could not be expressed in terms of less than four scalar functions, properties analogous to those for isotropic turbulence could not be derived. Chandrasekhar [10] expressed the second-order tensor B_{ij} explicitly in terms of only two scalar functions (Q_1 and Q_2). He introduced a new aspect of the theory which Batchelor did not consider: the representation of the axisymmetric solenoidal tensor as the curl of a general axisymmetric skew tensor. Chandrasekhar extended his method to the third-order tensor (two-point triple correlations). This tensor depends on six scalar independent functions. In this manner, a new "von Kármán-Howarth equation" was established for axisymmetric turbulence. However, Chandrasekhar's representation is complicated and not easy to be tested experimentally.

An interesting theoretical representation of axisymmetric turbulence was recently presented by Lindborg [11]. Axisymmetric tensors are expressed in terms of scalar functions corresponding to correlations which can be measured. This was not possible for the representations of Batchelor and Chandrasekhar. Unlike Batchelor and Chandrasekhar, Lindborg included rotational states (reflectional symmetry is imposed only in planes normal to the axis of symmetry; no symmetry is required for planes through the axis of symmetry). Skew tensors were introduced to describe mean flow rotation; they are zero when there is no rotation about the axis of symmetry. Lindborg obtained expressions for second and third-order axisymmetric two-point correlation tensors in terms of measurable scalar functions.

The approach used in the present paper follows that of Lindborg [11]. In particular, the "axisymmetric" equivalent of Karman-Howarth "isotropic" equation is derived in Section 5. The axisymmetric forms of the second and third-order tensors for the velocity correlations are given in Sections 3 and 4 respectively. There are two interesting limiting forms of the axisymmetric K-H equation.

2 Karman-Howarth equation

Departing from the Navier-Stokes equations, Monin has considered the derivation of the dynamic equation for the tensor $B_{ij}(\mathbf{r},t)$. This basic equation relates the second-order velocity correlation tensor $B_{ij}(\mathbf{r},t)$ to the third-order velocity correlation tensor $B_{ij,k}(\mathbf{r},t)$ for homogeneous turbulence. To find this equation, one can write Navier-Stokes equations for the ith velocity component (u_{0i}) at point $\mathbf{x_0}$, and the jth velocity component (u_j) at the point $\mathbf{x} = \mathbf{x_0} + \mathbf{r}$, and multiply the first of them by u_j and the second by u_{0i}. Then, add both equations together and take an average

$$\frac{\partial B_{ij}(\mathbf{r})}{\partial t} = T_{ij}(\mathbf{r}) + \Pi_{ij}(\mathbf{r}) - 2\epsilon_{ij}(\mathbf{r}), \tag{2}$$

and

$$\begin{aligned} B_{ij}(\mathbf{r}) &= \langle u_{0i} u_j\rangle, \\ B_{ij,k}(\mathbf{r}) &= \langle u_{0i} u_{0j} u_k\rangle \\ T_{ij}(\mathbf{r}) &= \frac{\partial}{\partial r_k}\left(B_{ik,j}(\mathbf{r}) - B_{i,jk}(\mathbf{r})\right) \\ \Pi_{ij}(\mathbf{r}) &= \frac{1}{\rho}\left(\frac{\partial\langle p_0 u_j\rangle}{\partial r_i} - \frac{\partial\langle p u_{0i}\rangle}{\partial r_j}\right) \\ \epsilon_{ij}(\mathbf{r}) &= -\nu\frac{\partial^2}{\partial r_k^2} B_{ij}(\mathbf{r}). \end{aligned}$$

Note that u_i (or p) depends only on $\mathbf{x}$ and u_{0i} (or p_0) depends only on $\mathbf{x_0}$. In this paper, repeated indices implies summation and no summation is implied by repeated Greek indices.

In equation (2), T_{ij} is the transfer tensor; its role is essentially to transfer energy from large to small scales. Π_{ij} is the two-point pressure strain tensor which vanishes for $i = j$; this correlation contributes nothing to the decay of kinetic energy, which is affected solely by viscosity, but is responsible for a transfer of energy between the longitudinal and transverse velocity components. And ϵ is the two-point dissipation tensor.

In the case of isotropic turbulence, the non-vanishing scalar functions of second and third-order ($B_{\rm NN}$, $B_{\rm LN,N}$ and $B_{\rm NN,L}$) can be expressed in terms of the two longitudinal scalar functions $B_{\rm LL}(r,t)$ and $B_{\rm LL,L}(r,t)$. Therefore, Monin and Yaglom obtained an equation which plays a basic part in all subsequent studies in the theory of isotropic turbulence:

$$\frac{\partial}{\partial t}B_{\rm LL}(r,t)=\left(\frac{\partial}{\partial r}+\frac{4}{r}\right)\left(B_{\rm LL,L}+2\nu\frac{\partial}{\partial r}B_{\rm LL}\right). \quad (3)$$

This equation was first derived by Karman and Howarth (1938) as pointed out Monin and Yaglom [2]. The left-hand-side of equation (3) is the temporal derivative of the two point correlation. The first term in the right-hand-side is the spatial derivative of the third-order correlation. It results from the non-linearity of the Navier-Stokes equations. The last term represents the effects of the molecular viscosity. This single scalar equation, for isotropic turbulence, yields several auxiliary equations of physical significance. Expanding the scalar functions in powers of r^2, and equating coefficients of similar powers of r^2, there is obtained, in turn, equations for the time rate of change of the mean square velocity fluctuation (the energy balance equation), of the mean square vorticity (the vorticity balance equation), etc.

In Section 5, we will derive the axisymmetric form of Karman-Howarth equation (K-H equation) and its limiting forms. Let us first consider equation (2). After contracting indices ($i=j$), it reduces to

$$\begin{aligned}\frac{\partial}{\partial t}B_{ii}(\mathbf{r},t)=&\frac{\partial}{\partial r_k}\left[B_{ik,i}(\mathbf{r},t)-B_{i,ik}(\mathbf{r},t)\right]\\&+2\nu\frac{\partial^2}{\partial r_k^2}B_{ii}(\mathbf{r},t),\end{aligned} \quad (4)$$

where the derivatives of pressure-velocity correlations, $\partial(B_{pi}-B_{ip})/\partial r_i$, vanish because of incompressibility (continuity). The case $i=j$ is very interesting because the difficulties introduced by the pressure-velocity correlations disappear. Homogeneity means that $B_{ij,k}(\mathbf{r},t)$ is independent of position $\mathbf{x}$ and also implies:

$$B_{i,ik}(\mathbf{r},t)=B_{ik,i}(-\mathbf{r},t). \quad (5)$$

Equation (4) is valid for homogeneous turbulence. When local isotropy is assumed, one can obtain K-H equation (3) from equation (4). By substituting the axisymmetric forms of B_{ii} and $B_{ik,i}$ in equation (4), we will derive the axisymmetric form of K-H equation and their limiting forms. One of them is the energy balance equation for axisymmetric turbulence; the other must be the axisymmetric form of the vorticity balance equation.

Note that in the present work, we have assumed in a first stage, that we are concerned with an incompressible fluid whose motion can be described by the Navier-Stokes equations without external forces, equation (2) (see M-Y [2]). Usually, inhomogeneities (mean gradients or shears) are needed to produce turbulence. In this case, we have to consider the effects of the mean flow gradient. Thus the dynamical equation for the second-order tensor B_{ij} can be written (see Lindborg [11] and Hinze [12])

$$\frac{\partial B_{ij}(\mathbf{r})}{\partial t}=-\frac{\partial U_k}{\partial x_s}A_{ksij}(\mathbf{r})+T_{ij}(\mathbf{r})+\Pi_{ij}(\mathbf{r})-2\epsilon_{ij}(\mathbf{r}),$$

where $\partial U_k/\partial x_s$ is the mean flow gradient tensor, and

$$A_{ksij}(\mathbf{r})=\delta_{ki}B_{sj}(\mathbf{r})+\delta_{kj}B_{si}(-\mathbf{r})+r_s\frac{\partial}{\partial r_k}B_{ij}(\mathbf{r}).$$

Frisch [13] added a forcing term $f(\mathbf{r},t)$ in the Navier-Stokes equations which is active only at large scales. The force f is needed to replenish the energy dissipated by viscosity.

However, if there is a production of turbulence by mean gradients in some regions of some flows, this energy can also be transferred by turbulent diffusion to other regions where the gradients vanish. This is the case of flows which present a symmetry in the preferred direction (for example: a jet, a chanal, a wake, ...). Note that the inhomogeneities may be weak enough to be partially ignored at small scales and far from boundaries, Frisch [13]. For these cases, our derivation is valid.

3 Second-order tensor B_{ij}

A system of orthogonal unit vectors $(\lambda,\mathbf{e}_1,\mathbf{e}_2)$ is chosen to represent axisymmetric tensors for velocity structure functions (see Fig. 1, which is similar to Fig. 1 of Lindborg [11]). In this system, the second-order axisymmetric tensor depends only on four independent scalar functions. For axisymmetric turbulence without mean rotation (only this case is considered here for simplicity), an important property of the second-order correlation function tensor is the symmetry index : $B_{ij}(-\mathbf{r})=B_{ij}(\mathbf{r})$ which allows the five scalar functions to be reduced to four, *viz.*

$$\begin{aligned}B_{ij}(\mathbf{r})=&\lambda_i\lambda_jB_1+e_{2i}e_{2j}B_2+e_{1i}e_{1j}B_3\\&+(\lambda_ie_{2j}+\lambda_je_{2i})B_4.\end{aligned} \quad (6)$$

B_1, B_2, B_3 and B_4, which depend on $\rho=|\mathbf{r}\times\lambda|$ and $z=\mathbf{r}\cdot\lambda$, can all be measured since

$$B_1=\langle u_{p1}u_{01}\rangle \quad (7a)$$
$$B_2=\langle u_{p2}u_{02}\rangle \quad (7b)$$
$$B_3=\langle u_{p3}u_{03}\rangle \quad (7c)$$
$$B_4=\langle u_{01}u_{p2}\rangle. \quad (7d)$$

Also, B_1, B_2, B_3 are even in z while B_4 is odd with respect to z. (As can be seen in Fig. 1, the image of point P with respect to a plane perpendicular to vector λ, is such that $u_1'=-u_1$, $u_2'=u_2$, $u_3'=u_3$).

As noted by Lindborg, the separation vector $\mathbf{r}$ and all the velocity components in a plane normal to λ can be reflected with respect to this plane without affecting $D_{ij}(\mathbf{r})$. Reflectional symmetry about a plane containing the axis of symmetry, for example the $(\lambda,\mathbf{r})$ plane, implies

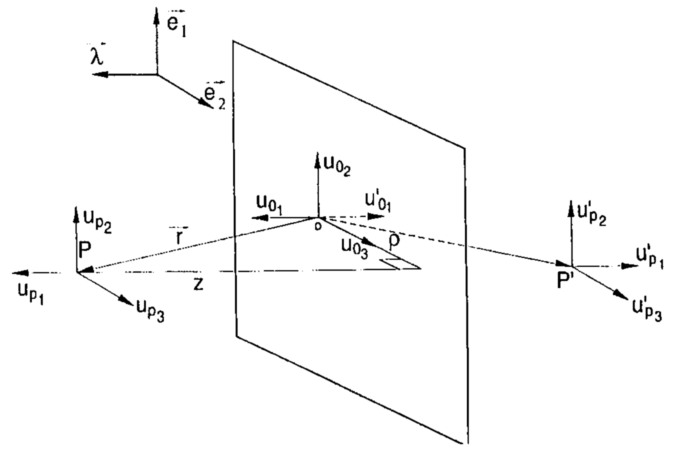

Fig. 1. Cartesian co-ordinate system showing velocity components.

that the skew tensors are zero; this is true when there is no rotation about the axis of symmetry.

Relations between B_1, B_2, B_3, B_4 follow from continuity, *i.e.*

$$\frac{\partial}{\partial r_j} B_{ij}(\mathbf{r}) = \frac{\partial}{\partial r_i} B_{ij}(\mathbf{r}) = 0. \tag{8}$$

The procedure is similar to that established by Lindborg and yields equivalent results

$$\frac{\partial}{\partial \rho}(\rho B_4) + \rho \frac{\partial}{\partial z}(B_1) = 0 \tag{9a}$$

$$B_3 = \frac{\partial}{\partial \rho}(\rho B_2) + \rho \frac{\partial}{\partial z} B_4 \tag{9b}$$

where we have used the following derivatives with respect to r_j

$$\frac{\partial}{\partial r_j} e_{1i} = -\frac{1}{\rho} e_{1j} e_{2i} \quad , \qquad \frac{\partial}{\partial r_j} e_{2i} = \frac{1}{\rho} e_{1j} e_{1i} \tag{10a}$$

$$\lambda_j \frac{\partial}{\partial r_j} = \frac{\partial}{\partial z} \quad , \qquad e_{2j} \frac{\partial}{\partial r_j} = \frac{\partial}{\partial \rho},$$

$$e_{1j} \frac{\partial}{\partial r_j} = 0. \tag{10b}$$

For isotropic turbulence, λ can lie in any direction. Before checking that these results are compatible with the well-known isotropic relation between $B_{\rm LL}(r)$ and $B_{\rm NN}(r)$, it should be noted that derivatives with respect to z or ρ are linked to those with respect to r and/or μ ($\mu = \mathbf{r} \cdot \lambda / r$) as follows

$$\frac{\partial}{\partial z} = \lambda_j \frac{\partial}{\partial r_j} = r\mu D_r + D_\mu \tag{11a}$$

$$\frac{\partial}{\partial \rho} = e_{2j} \frac{\partial}{\partial r_j} = r\sqrt{1-\mu^2} D_r \tag{11b}$$

with

$$\left. \begin{aligned} \frac{\partial}{\partial r_j} &= r_j D_r + \lambda_j D_\mu \\ D_r &= \frac{1}{r}\frac{\partial}{\partial r} - \frac{\mu}{r^2}\frac{\partial}{\partial \mu} \\ D_\mu &= \frac{1}{r}\frac{\partial}{\partial \mu} \end{aligned} \right\} .$$

In order to relate B_1, B_2, B_3, B_4 to $B_{\rm LL}$ and $B_{\rm NN}$, the second-order isotropic tensor

$$B_{ij}(\mathbf{r}) = (B_{\rm LL} - B_{\rm NN}) \frac{r_i r_j}{r^2} + B_{\rm NN} \delta_{ij} \tag{12}$$

is projected onto the tensors $\lambda_i \lambda_j$, $e_{1i} e_{1j}$, $e_{2i} e_{2j}$ and $\lambda_i e_{2j}$ corresponding to B_1, B_2, B_3 and B_4 respectively. Since it is also known that

$$e_{1i} e_{1j} = \delta_{ij} - \lambda_i \lambda_j - e_{2i} e_{2j} \tag{13a}$$

$$e_{2i} = \frac{1}{r(1-\mu^2)^{1/2}} (r_i - r\mu\lambda_i), \tag{13b}$$

the following relations can be derived

$$B_1 = \frac{z^2}{r^2} B_{\rm LL} + \frac{\rho^2}{r^2} B_{\rm NN} \tag{14a}$$

$$B_2 = \frac{\rho^2}{r^2} B_{\rm LL} + \frac{z^2}{r^2} B_{\rm NN} \tag{14b}$$

$$B_3 = B_{\rm NN} \tag{14c}$$

$$B_4 = \frac{\rho z}{r^2} (B_{\rm LL} - B_{\rm NN}). \tag{14d}$$

After substituting (14a-d) into equations (9a, b) and taking into account the equivalences (11a, b), the isotropic result

$$B_{\rm NN}(r) = \left(1 + \frac{r}{2}\frac{\partial}{\partial r}\right) B_{\rm LL}(r) \tag{15}$$

is obtained. Note that when $\mathbf{r}$ is parallel to λ, $\rho = 0$, $B_2 = B_3$ and $B_4 = 0$; this leads to

$$B_{ij}(\mathbf{r}) = \lambda_i \lambda_j B_1 + B_2 (\delta_{ij} - \lambda_i \lambda_j) \tag{16}$$

which is similar in form to (12). However, whereas $B_{\rm LL}$ and $B_{\rm NN}$ are related through (15), B_1 and B_2 are independent scalar functions.

4 Third-order tensor $B_{ij,k}$

In this section, we develop the two-point representation for the third-order tensors and establish the properties of the corresponding scalar functions that are needed for reduction of the general formulae in Sections 5. The third-order correlation tensor, $B_{ij,k}(\mathbf{r}, t)$, is symmetric in the first two indices (i and j) and can be represented, in the case of axisymmetric turbulence without mean rotation about the axis of symmetry:

$$\begin{aligned} B_{ij,k}(\mathbf{r}) = {} & \lambda_i\lambda_j\lambda_k M_1 + \lambda_k e_{2i}e_{2j}M_2 \\ & + \lambda_k e_{1i}e_{1j}M_3 + \lambda_i\lambda_j e_{2k}M_4 + e_{2i}e_{2j}e_{2k}M_5 \\ & + e_{1i}e_{1j}e_{2k}M_6 + (\lambda_i e_{2j} + \lambda_j e_{2i})\lambda_k M_7 \\ & + (\lambda_i e_{2j}+\lambda_j e_{2i})e_{2k}M_8 + (\lambda_i e_{1j}+\lambda_j e_{1i})e_{1k}M_9 \\ & + (e_{2i}e_{1j} + e_{2j}e_{1i})e_{1k}M_{10} \end{aligned} \quad (17)$$

where M_1, $M_2 \cdots M_{10}$ are scalar functions of ρ and z. M_4, M_5, M_6, M_7 and M_{10} are even in z, while the rest of scalar functions are odd in z. These functions are defined as follow

$$\begin{aligned} M_1 &= \langle u_{01}^2 u_{p1}\rangle, & M_2 &= \langle u_{02}^2 u_{p1}\rangle, \\ M_3 &= \langle u_{03}^2 u_{p1}\rangle, & M_4 &= \langle u_{01}^2 u_{p2}\rangle, \\ M_5 &= \langle u_{02}^2 u_{p2}\rangle, & M_6 &= \langle u_{03}^2 u_{p2}\rangle, \\ M_7 &= \langle u_{01}u_{02}u_{p1}\rangle, & M_8 &= \langle u_{01}u_{02}u_{p2}\rangle, \\ M_9 &= \langle u_{01}u_{03}u_{p3}\rangle, & M_{10} &= \langle u_{03}u_{02}u_{p3}\rangle. \end{aligned} \quad (18)$$

Moreover, $B_{ij,k}$ is solenoidal in the last index

$$\frac{\partial}{\partial r_k} B_{ij,k} = 0. \quad (19)$$

Using the general expression of $B_{ij,k}$, equation (17), and relations (11a, b), the continuity condition, (Eq. (19)), leads to relations between the ten scalar functions

$$\frac{\partial}{\partial\rho}(\rho M_4) + \rho\frac{\partial}{\partial z}(M_1) = 0 \quad (20a)$$

$$M_{10} = \frac{1}{2}\left(\frac{\partial}{\partial\rho}(\rho M_5) + \rho\frac{\partial}{\partial z}M_2\right) \quad (20b)$$

$$M_{10} = -\frac{1}{2}\left(\frac{\partial}{\partial\rho}(\rho M_6) + \rho\frac{\partial}{\partial z}M_3\right) \quad (20c)$$

$$M_9 = \frac{\partial}{\partial\rho}(\rho M_8) + \rho\frac{\partial}{\partial z}M_7. \quad (20d)$$

In order to find the relations between the axisymmetric scalar functions M_1, M_2, ..., M_{10} and the isotropic third-order scalar functions $B_{\mathrm{LL,L}}$, $B_{\mathrm{LN,N}}$ and $B_{\mathrm{NN,L}}$, let us consider the third-order isotropic tensor which can be expressed in terms of the three non-vanishing isotropic scalar functions

$$\begin{aligned} B_{ij,k}(\mathbf{r}) = {} & (B_{\mathrm{LL,L}} - 2B_{\mathrm{LN,N}} - B_{\mathrm{NN,L}})\frac{r_i r_j r_k}{r^3} \\ & + B_{\mathrm{LN,N}}\left(\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik}\right) + B_{\mathrm{NN,L}}\frac{r_k}{r}\delta_{ij}. \end{aligned} \quad (21)$$

The projection of the isotropic tensor $B_{ij,k}$ onto the $\lambda_i\lambda_j\lambda_k$, $\lambda_k e_{2i}e_{2j}$, $\lambda_k e_{1i}e_{1j}$, $\lambda_i\lambda_j e_{1k}$, $e_{2i}e_{2j}e_{2k}$, $e_{1i}e_{1j}e_{2k}$, $\lambda_i\lambda_k e_{2k}$, $\lambda_i e_{2j}e_{2k}$, $\lambda_i e_{1j}e_{1k}$ and $e_{2i}e_{1j}e_{1k}$ which corresponds to M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9 and M_{10} respectively, gives the relations between the axisymmetric functions and the isotropic ones

$$M_1 = (B_{\mathrm{LL,L}} - 2B_{\mathrm{LN,N}} - B_{\mathrm{NN,L}})\frac{z^3}{r^3} + 2B_{\mathrm{LN,N}}\frac{z}{r} + B_{\mathrm{NN,L}}\frac{z}{r} \quad (22a)$$

$$M_2 = (B_{\mathrm{LL,L}} - 2B_{\mathrm{LN,N}} - B_{\mathrm{NN,L}})\frac{z\rho^2}{r^3} + B_{\mathrm{NN,L}}\frac{z}{r} \quad (22b)$$

$$M_3 = B_{\mathrm{NN,L}}\frac{z}{r} \quad (22c)$$

$$M_4 = (B_{\mathrm{LL,L}} - 2B_{\mathrm{LN,N}} - B_{\mathrm{NN,L}})\frac{\rho z^2}{r^3} + B_{\mathrm{NN,L}}\frac{\rho}{r} \quad (22d)$$

$$M_5 = (B_{\mathrm{LL,L}} - 2B_{\mathrm{LN,N}} - B_{\mathrm{NN,L}})\frac{\rho^3}{r^3} + 2B_{\mathrm{LN,N}}\frac{\rho}{r} + B_{\mathrm{NN,L}}\frac{\rho}{r} \quad (22e)$$

$$M_6 = B_{\mathrm{NN,L}}\frac{\rho}{r} \quad (22f)$$

$$M_7 = (B_{\mathrm{LL,L}} - 2B_{\mathrm{LN,N}} - B_{\mathrm{NN,L}})\frac{\rho z^2}{r^3} + B_{\mathrm{LN,L}}\frac{\rho}{r} \quad (22g)$$

$$M_8 = (B_{\mathrm{LL,L}} - 2B_{\mathrm{LN,N}} - B_{\mathrm{NN,L}})\frac{z\rho^2}{r^3} + B_{\mathrm{LN,L}}\frac{z}{r} \quad (22h)$$

$$M_9 = B_{\mathrm{LN,L}}\frac{z}{r} \quad (22i)$$

$$M_{10} = B_{\mathrm{LN,L}}\frac{\rho}{r}\,\cdot \quad (22j)$$

In the special case of $\mathbf{r}$ parallel to λ, $M_2 = M_3$, $M_8 = M_9$ and $M_4 = M_5 = M_6 = M_7 = M_{10} = 0$; this leads to

$$\begin{aligned} B_{ij,k}(\mathbf{r}) = {} & (M_1 - 2M_8 - M_2)\lambda_i\lambda_j\lambda_k \\ & + M_8(\lambda_i\delta_{jk} + \lambda_j\delta_{ik}) + M_2\lambda_k\delta_{ij}, \end{aligned} \quad (23)$$

which is similar to the isotropic expression of $B_{ij,k}$, (Eq. (21)). However, the scalar isotropic functions $B_{\mathrm{LL,L}}$, $B_{\mathrm{LN,N}}$ and $B_{\mathrm{NN,L}}$ are related whereas the remaining scalar functions, M_1, M_2 and M_8, are independent.

After substituting equations (22a-j) into equations (20a-d), we obtain the well-known relations between the isotropic scalar functions of the tensor $B_{ij,k}$

$$B_{\mathrm{NN,L}} = -\frac{1}{2}B_{\mathrm{LL,L}} \quad (24)$$

$$B_{\mathrm{LN,N}} = \frac{1}{2}\left(1 + \frac{r}{2}\frac{\partial}{\partial r}\right)B_{\mathrm{LL,L}}. \quad (25)$$

5 Axisymmetric form of “Karman-Howarth” equation

The purpose of the present section is, first, to establish the axisymmetric form of the dynamic equation. Then, we derive the corresponding limiting form when axisymmetry turns into isotropy, on the one hand, and the limiting forms when the separation r goes to zero, on the other. To derive the axisymmetric form of Karman-Howarth equation, we consider equation (4) which was established for homogeneous turbulence. When we substitute the axisymmetric expressions of the tensors B_{ii} and $\mathrm{B}_{i\alpha,i}$ into this equation, we obtain

$$\frac{\partial}{\partial t}\sum_{i=1}^{3} B_i = \frac{\partial}{\partial z}\sum_{i=1,8,9}(M_i - M_i^*) + \left(\frac{1}{\rho}+\frac{\partial}{\partial\rho}\right)\sum_{i=5,7,10}(M_i - M_i^*) + 2\nu\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\sum_{i=1}^{3} B_i \qquad (26)$$

where

$$\sum_{i=1,8,9} M_i = M_1 + M_8 + M_9.$$

The M^*'s are the scalar functions relative to the tensor $B_{i,ik}(\mathbf{r},t)$

$$M_1^* = \langle u_{p1}^2 u_{01}\rangle, \qquad M_5^* = \langle u_{p2}^2 u_{02}\rangle,$$
$$M_7^* = \langle u_{01} u_{p1} u_{p2}\rangle, \qquad M_8^* = \langle u_{02} u_{p1} u_{p2}\rangle,$$
$$M_9^* = \langle u_{03} u_{p1} u_{p3}\rangle, \qquad M_{10}^* = \langle u_{03} u_{p2} u_{p3}\rangle.$$

Note that it is also possible to work with the tensor $B_{ik,i}(-\mathbf{r},t)$ instead of $B_{i,ik}(\mathbf{r},t)$ as for homogeneous turbulence $B_{ik,i}(-\mathbf{r},t) = B_{i,ik}(\mathbf{r},t)$

Spectral form of equation (26):

The spectral form of K-H equation has a simple physical interpretation which is important for the understanding of the mechanism of turbulent mixing. It is more convenient to derive this spectral form from equation (2) (or Eq. (4) with $i \neq j$) which was used to derive equation (26). For any homogeneous turbulence, the Fourier-transform of each term in equation (2) leads to

$$\frac{\partial \hat{B}_{ij}(\mathbf{k})}{\partial t} = \hat{T}_{ij}(\mathbf{k}) + \hat{\Pi}_{ij}(\mathbf{k}) - 2\nu k^2 \hat{B}_{ij}(\mathbf{k}), \qquad (27)$$

where $\hat{T}_{ij}(\mathbf{k})$ is the Fourier transform (F-T) of the transfer tensor $T_{ij}(\mathbf{r})$, $\hat{\Pi}_{ij}(\mathbf{k})$ is the F-T of the pressure-strain tensor $\Pi_{ij}(\mathbf{r})$ and $\nu k^2 \hat{B}_{ij}(\mathbf{k})$ the F-T of the viscous dissipation $\epsilon_{ij}(\mathbf{r})$. Substituting $i=j$ in the previous equation, one can obtain

$$\frac{\partial \hat{B}_{ii}(\mathbf{k})}{\partial t} = \hat{T}_{ii}(\mathbf{k}) - 2\nu k^2 \hat{B}_{ii}(\mathbf{k}). \qquad (28)$$

The left-hand-side of this equation describes the time variation of the turbulent energy with the wave number k. The corresponding components of $\hat{T}_{ii}$ describes the non linear transfer of energy within each component. This transfer is just an energy redistribution among the individual spectral components, without any change in the energy of the turbulent motion which is conserved, Lindborg [11] and M-Y [2]. The viscous energy dissipation is described by the second-term on the right-hand-side. It causes a decrease in the kinetic energy. For small values of k, this energy decrease is much more slower than the energy decrease for high values of k.

5.1 Isotropy

The axisymmetric result (Eq. (26)) should reduce to the isotropic form when λ is allowed to assume any direction. In this case, we can put the expressions of the M-functions given in (22a-j) and the expressions of the B-functions given in (14a-d) as well as relations (between the scalar functions $B_{\mathrm{LL,L}}$, $B_{\mathrm{LN,N}}$ and $B_{\mathrm{NN,L}}$) (24) and (25) into equation (26). Note that, for isotropic turbulence, we have $M_i^* = -M_i$, and $B_{i,ik}(\mathbf{r},t) = -B_{ik,i}(\mathbf{r},t)$. Therefore, using (11 a-c), we find

$$\left(3 + r\frac{\partial}{\partial r}\right)\left\{\frac{\partial B_{\mathrm{LL}}}{\partial t} - \left(\frac{4}{r}+\frac{\partial}{\partial r}\right)\left(B_{\mathrm{LL,L}} + 2\nu\frac{\partial B_{LL}}{\partial r}\right)\right\} = 0. \qquad (29)$$

Using Monin-Yaglom's argument (p. 122), the only solution of equation (29) which does not have a singularity at $r=0$ is given by K-H equation (3).

Finally, it is worth noting that, in the particular axisymmetric case when $\mathbf{r}$ is parallel to λ, equation (26) can be somewhat simplified. As mentioned before, $B_2 = B_3$, $M_8 = M_9$ and $M_5 = M_7 = M_{10} = 0$. However, we have to be careful at this stage: although M_5, M_7 and M_{10} are zero, their derivative with respect to ρ and their division by ρ are not equal to zero.

5.2 Limiting form when r → 0

5.2.1 Second-order functions

For small values of r, the form of the second-order correlation functions, B_1, B_2, B_3 and B_4, can be deduced from the Taylor expansions as far as the fourth power of r. For $\alpha = 1, 2$ or 3, the general form of these expansions is

$$B_\alpha = \langle u_\alpha^2\rangle + \frac{\rho^2}{2}\langle u_\alpha \frac{\partial^2 u_\alpha}{\partial\rho^2}\rangle + \frac{z^2}{2}\langle u_\alpha \frac{\partial^2 u_\alpha}{\partial z^2}\rangle + \frac{\rho^4}{24}\langle u_\alpha \frac{\partial^4 u_\alpha}{\partial\rho^4}\rangle + \frac{z^4}{24}\langle u_\alpha \frac{\partial^4 u_\alpha}{\partial z^4}\rangle + \frac{\rho^2 z^2}{4}\langle u_\alpha \frac{\partial^4 u_\alpha}{\partial\rho^2\partial z^2}\rangle \qquad (30)$$

and

$$B_4 = \rho z\langle u_1 \frac{\partial^2 u_2}{\partial\rho\partial z}\rangle + \frac{\rho^3 z}{6}\langle u_1 \frac{\partial^4 u_2}{\partial\rho^3\partial z}\rangle + \frac{\rho z^3}{6}\langle u_1 \frac{\partial^4 u_2}{\partial\rho\partial z^3}\rangle \qquad (31)$$

or in a simple manner

$$B_\alpha = B_{0\alpha} + a_\alpha\rho^2 + b_\alpha z^2 + c_\alpha\rho^4 + d_\alpha z^4 + e_\alpha\rho^2 z^2 + ... \qquad (32)$$

and

$$B_4 = a_4 z\rho + b_4 z\rho^3 + c_4 z^3\rho + ... \qquad (33)$$

The continuity equations (9a, b) allow us to reduce the number of coefficients in the previous expansions. For the particular case when $\mathbf{r}$ is parallel to λ, we obtain

$$\begin{aligned} &a_4 = -b_1, \quad b_4 = -\frac{1}{2}e_1, \quad c_4 = -2d_1, \\ &b_2 = b_3, \quad d_2 = d_3, \quad a_3 = 3a_2 + a_4, \\ &c_3 = 5c_2 + b_4, \quad e_3 = 3e_2 + 3c_4. \end{aligned} \tag{34}$$

Similarly, we can write the expansions of the isotropic functions f and g

$$f = \frac{B_{\rm LL}(r)}{B_{\rm LL}(0)} = 1 - \frac{r^2}{2\lambda_{\rm t}^2} + \frac{\alpha}{4!}r^4 + ... \tag{35}$$

$$g = \frac{B_{\rm NN}(r)}{B_{\rm NN}(0)} = 1 - \frac{r^2}{\lambda_{\rm t}^2} + \frac{3\alpha}{4!}r^4 + ... \tag{36}$$

where $\lambda_{\rm t}$ is the Taylor microscale and $B_{\rm LL}(0) = B_{\rm NN}(0) = u^2$, u being the root mean square of any velocity component. Now, when axisymmetry turns into isotropy, equations (14a-d) yields to

$$\begin{aligned} &a_1 = 2a_2 = a_3 = 2b_1 = b_2 = b_3 = -\frac{u^2}{\lambda_t^2} \\ &c_1 = 3c_2 = c_3 = 3d_1 = d_2 = d_3 = \frac{\alpha}{8}u^2 \\ &6e_1 = 6e_2 = 4e_3 = \alpha u^2. \end{aligned} \tag{37}$$

5.2.2 Third-order functions

The behaviour of the third-order correlation functions, for small separations ($r \to 0$), are given by the following expansions for $i = 1, 2, 3, 8$ or 9

$$M_i = \frac{z^3}{6}\langle u_1 u_\alpha \frac{\partial^3 u_\alpha}{\partial z^3}\rangle + \frac{z\rho^2}{2}\langle u_1 u_\alpha \frac{\partial^3 u_\alpha}{\partial \rho^2 \partial z}\rangle \tag{38}$$

where for i = 1, 8 or 9, it corresponds α = 1, 2 or 3 respectively. We can equivalently write M_i in a simple manner

$$M_i = \alpha_i z^3 + \beta_i z\rho^2 + ... \tag{39}$$

Moreover, for i = 4, 5, 6, 7 or 10

$$M_i = \frac{\rho^3}{6}\langle u_2 u_\alpha \frac{\partial^3 u_\alpha}{\partial \rho^3}\rangle + \frac{\rho z^2}{2}\langle u_2 u_\alpha \frac{\partial^3 u_\alpha}{\partial \rho \partial z^2}\rangle \tag{40}$$

where α = 1, 2 or 3 is relative to i = 5,7 or 10 respectively. We can also simply write

$$M_i = \gamma_i \rho^3 + \zeta_i z^2 \rho + ... \tag{41}$$

From the continuity equations (20a-d), the coefficients α_i, β_i γ_i and ζ_i are required to satisfy

$$\begin{aligned} &\beta_1 = -4\gamma_4, \quad \alpha_1 = -\frac{2}{3}\zeta_4, \quad \alpha_9 = \alpha_8, \\ &\gamma_{10} = 2\gamma_5 + \frac{1}{2}\beta_2 = -2\gamma_6 - \frac{1}{2}\beta_3, \\ &\zeta_{10} = \zeta_5 + \frac{3}{2}\alpha_2 = -\zeta_6 - \frac{3}{2}\alpha_3, \\ &\beta_9 = 3\beta_8 + 2\zeta_7. \end{aligned} \tag{42}$$

Moreover, in the particular case $\rho = 0$, we have $\alpha_2 = \alpha_3$.

For small values of r, the longitudinal scalar function $B_{\rm LL,L}$ is

$$B_{\rm LL,L}(r) = b_3 r^3 + ... \tag{43}$$

where

$$b_3 = \tau u^3 = \frac{1}{6}\left[\frac{\partial^3 B_{\rm LL,L}(r)}{\partial r^3}\right]_{r=0} = \frac{1}{6}\langle\left(\frac{\partial u_1}{\partial x_1}\right)^3\rangle. \tag{44}$$

Finally, for isotropy, combination of equations (24), (25) and (43) gives the relations between the coefficients in (22a-j) and the single coefficient b_3,

$$\begin{aligned} &\alpha_1 = \gamma_5 = b_3, \quad \beta_8 = \zeta_7 = \frac{1}{4}b_3, \\ &\alpha_9 = \alpha_8 = \beta_9 = \gamma_7 = \gamma_{10} = \zeta_{10} = \frac{5}{4}b_3, \\ &\beta_1 = \zeta_5 = 2b_3. \end{aligned} \tag{45}$$

5.3 The zero-order terms of Taylor expansions

There are two interesting limiting forms of the axisymmetric form of K-H equation. When we consider only the zero-order terms of Taylor expansions in equation (26), we can deduce one limiting form which is, in fact, an axisymmetric form of the energy balance equation. Furthermore, when we focus on the second-order terms of Taylor expansions, equation (26) seems to be reduced to the axisymmetric form of the vorticity balance equation.

It is straightforward to derive the limiting form of the axisymmetric form of Karman-Howarth equation (26) using the previous expansions of the second and third-order correlation functions as far as the second power of r (*i.e.* of ρ and z)

$$\frac{\partial}{\partial t}\sum_{i=1}^{3} B_{0i} = 2\nu\left[2\sum_{i=1}^{3} b_i + 4\sum_{i=1}^{3} a_i\right] \tag{46}$$

or

$$\frac{\partial}{\partial t}\langle u_i^2\rangle = 2\nu(\langle u_i \frac{\partial^2 u_i}{\partial z^2}\rangle + 2\langle u_i \frac{\partial^2 u_i}{\partial \rho^2}\rangle). \tag{47}$$

Note that the axisymmetric form of the viscous dissipation, $\epsilon_{\rm axi}$ is

$$\epsilon_{\rm axi} = -\nu\left[\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right) B_{ii}(\mathbf{r})\right]_{r=0} \tag{48}$$

where $B_{ii} = B_1 + B_2 + B_3$. By using Taylor expansions for B_1, B_2 and B_3 (up to the second-order powers of r) into equation (48), we find

$$\epsilon_{\rm axi} = -4\nu\sum_{i=1}^{3} a_i - 2\nu\sum_{i=1}^{3} b_i \tag{49}$$

and for $i = 1, 2, 3$

$$a_i = \frac{1}{2!}\left(\frac{\partial^2 B_i}{\partial \rho^2}\right)_{r=0}$$
$$b_i = \frac{1}{2!}\left(\frac{\partial^2 B_i}{\partial z^2}\right)_{r=0} \cdot$$

Therefore, the limiting form (46) is simply written

$$\frac{\partial}{\partial t}\sum_{i=1}^{3} B_{0i} = -2\epsilon_{\text{axi}}. \tag{50}$$

When $\rho = 0$, it reduces to

$$\frac{\partial}{\partial t}(B_{01} + 2B_{02}) = -2\epsilon_{\text{axi}} \tag{51}$$

with

$$\frac{\epsilon_{\text{axi}}}{\nu} = -\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\rangle + 2\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\rangle + 8\langle\left(\frac{\partial u_2}{\partial x_2}\right)^2\rangle + 2\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\rangle \cdot \tag{52}$$

This is the energy balance equation for axisymmetric turbulence. It describes the rate of viscous decrease of the mean kinetic energy of turbulence. In the case of isotropy, equation (50) or equation (51) reduces to the well-known isotropic energy balance equation which was first derived by Taylor (1935, see M-Y [2]). Indeed, for isotropic turbulence, we have

$$\langle u_1^2\rangle = \langle u_2^2\rangle = \langle u_3^2\rangle \tag{53}$$

and with the help of relations (35) and (37), we deduce the well-known isotropic result

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{3}{2}\langle u_1^2\rangle) = -15\nu\frac{\langle u_1^2\rangle}{\lambda_{\mathrm{t}}^2} \cdot \tag{54}$$

One can readily show that equation (54) can be written in term of the isotropic viscous dissipation ($\epsilon_{\text{iso}} = 15\nu\langle u_{1,1}^2\rangle$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u_1^2\rangle = -2\epsilon_{\text{iso}} \tag{55}$$

which is similar to the axisymmetric result equations (50) or (51).

5.4 The second-order terms of Taylor expansions

Now, let us consider the Taylor expansions given in equations (32), (39) and (41) as far as the fourth power of r (*i.e.* ρ^4, z^4 and $z^2\rho^2$). Substituting them in the axisymmetric form of K-H equation (26), we obtain

$$\frac{\partial}{\partial t}\left(\rho^2\sum_{i=1}^{3} a_i + z^2\sum_{i=1}^{3} b_i\right) = \nu\rho^2\left(32\sum_{i=1}^{3} c_i + 4\sum_{i=1}^{3} e_i\right) + \nu z^2\left(8\sum_{i=1}^{3} e_i + 24\sum_{i=1}^{3} d_i\right) + \rho^2\left(\sum_{i=1,8,9}(\beta_i - \beta_i^*) + 4\sum_{i=5,7,10}(\gamma_i - \gamma_i^*)\right) + z^2\left(3\sum_{i=1,8,9}(\alpha_i - \alpha_i^*) + 2\sum_{i=5,7,10}(\zeta_i - \zeta_i^*)\right). \tag{56}$$

This must be the axisymmetric form of the vorticity balance equation because it reduces to the well-known vorticity balance equation in the case of isotropic turbulence. Indeed, substituting relations (37), (43) and (45) into equation (56), we obtain

$$\frac{\partial}{\partial t}\left\{\left(-\frac{5}{2}\frac{u^2}{\lambda_t^2}\right)(\rho^2 + z^2)\right\} = 2\nu\{\frac{35}{6}\alpha u^2(\rho^2 + z^2)\} + 35\tau u^2(\rho^2 + z^2) \tag{57}$$

or

$$\frac{1}{2}\frac{\partial}{\partial t}B_{\text{LL}}^{(2)}(0) = \frac{7}{3}\nu B_{\text{LL}}^{(4)}(0) + \frac{7}{6}B_{\text{LL,L}}^{(3)}(0) \tag{58}$$

where

$$B_{\text{LL}}^{(2)}(0) = -\frac{u^2}{\lambda_t^2}, \qquad B_{\text{LL,L}}^{(3)}(0) = 6\tau u^3, \qquad B_{\text{LL}}^{(4)}(0) = \alpha u^2 \tag{59}$$

where $B^{(n)}(0)$ denotes the nth-order derivative with respect to r, for the value $r = 0$. Equation (58) is interpreted as the equation for the rate of change of the mean square vorticity $\langle\omega^2\rangle$, *i.e.* the vorticity balance equation (see M-Y [2]).

One of the primary objectives of this paper is to examine the statistics and the dynamics of fields which are homogeneous, but not isotropic or local isotropic. Furthermore, there is a considerable evidence that local isotropy is not an adequate description of the velocity derivatives moments for at least the finite Reynolds numbers associated with many turbulent laboratory flows (George and Hussein [14]). There are also few real turbulent flows in which the turbulence can be assumed to be isotropic. Only the statistical properties of the smallest scales of motion, in high-Reynolds-numbers flows, would be expected to satisfy isotropy. This strengthens considerably the suspicion that local isotropy is not a strong requirement for a correct description of many turbulent flows. In this context, experimental and numerical data for the results established in this paper obtained over a range of Reynolds numbers will be essential to resolve the questions raised above, in particular about why and whether or not local axisymmetry will persist as the Reynolds number is increased. A range of possibly different flows would be also desirable to show that the axisymmetric equations should have more general validity than their isotropic counterparts.

6 Conclusions

This new analysis offers some tools to investigators interested in the fundamental questions of turbulence. Kinematics and dynamics of homogeneous axisymmetric turbulence have been derived with the assumption that the properties of the turbulence are invariant with respect to rotation about a preferred direction λ. In particular, equation (26) which relates the third-order velocity correlation function to the second-order velocity correlation function. When axisymmetry turns into isotropy, equation (26) reduces to Karman-Howarth equation. When $r \to 0$, the limiting form of axisymmetric Karman-Howart equation leads to two limiting forms: the axisymmetric form of the energy balance equation and the axisymmetric form of the vorticity balance equation.

It is interesting to test experimentally as well as numericaly these new axisymmetric results, show their validity and determine whether or not the constrains of locally axisymmetic turbulence are satisfied. The development of such a theory of axisymmetric turbulence may also be useful in establishing the circumstances under which isotropy may be expected to prevail. A study of different flows with various Reynolds numbers may be provide a basis for discussing "why and whether or not local axisymmetry will persist as the turbulence Reynolds number is increased".

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